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THREE KINDS OF MATHEMATICAL FICTIONALISM*

> Dios sabe si hay Dulcinea o no en el mundo, o si es fantástica, o no es fantástica; y éstas no son de las cosas cuya averiguación se ha de llevar hasta el cabo. Ni yo engendré ni parí a mi señora, puesto que la contemplo como conviene que sea una dama que contenga en sí las partes que puedan hacerla famosa en todas las del mundo.
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> *Quijote, II, xxxii.***

Mario Bunge's *Ontology*[1] has revived the program of seventeenth century continental European philosophy for furthering the study of being *qua* being with the aid of exact mathematical concept building. Undeterred by Kant's warnings against the use of the mathematical method in philosophy,[2] Bunge feels free to resort to the entire established repertory of mathematical structures in his search for an adequate representation of the basic structures of reality. Abstract algebra turns out to be particularly helpful – a fact that will not surprise us if we recall that Leibniz's pioneering work in the field was done primarily in the service of ontology (and not merely of logic, as some twentieth-century commentators would have it).[3] But while in Leibniz's philosophy mathematics can be regarded as a part of ontology, for, as the abstract theory of *possibilia*, it spells out the general framework with which every theory of *realia* must agree, Bunge deliberately excludes mathematics from ontology and subscribes to the thesis that the subject matter of pure mathematics is totally fictitious [1974–, I, 27]. Bunge writes:

> We feign that there are constructs, i.e., creations of the human mind to be distinguished not only from things (e.g., words) but also from individual brain processes. (Only, we do not assume that constructs exist independently of brain processes.) We distinguish four basic kinds of constructs: concepts, propositions, contexts and theories. (...) Concepts are the building blocks of propositions. (...) A concept is either an individual (...) or a set (...) or a relation (...). The most interesting relations are the functions. (...) A context is a set of propositions sharing a reference class, and a theory is a context closed under the operation of deduction. [1974–, III, 116 f]

All constructs revolve therefore about propositions, of which Bunge says, in another passage:

J. Agassi and R.S. Cohen (eds.), Scientific Philosophy Today, 399–414.

> We do not claim that they exist in themselves but only that it is often convenient (for example in mathematics but not in metaphysics) to feign or pretend that they do. We do not assert that the Pythagorean theorem exists anywhere except in the world of phantasy called 'mathematics', a world that will go down with the last mathematician. [1974–, II, 85]

Thus Bunge, a staunch realist in the philosophy of physics, embraces fictionalism in the philosophy of mathematics. This combination is probably unprecedented, though Henri Poincaré may have been groping after it.[4] It is understandable in someone who, like Bunge, looks on Platonism as a "weird metaphysical hypothesis" [1974–, II, 85] and rejects or doubts the contention of objective idealism, that "concrete objects (things) have (...) intrinsic conceptual properties, in particular (...) mathematical features" [1974–, III, 118]. But mathematical fictionalism can be variously understood and, as I intend to show, not every conceivable version of it is equally at variance with Platonism. For greater precision I shall propose three interpretations of the thesis that mathematics is concerned with fictions. I shall then try to ascertain which of the three agrees better with the actual workings of mathematics, pure and applied. We shall see that the form of mathematical fictionalism that meets this condition, and which I take to be the one actually held by Bunge, does not differ from, say, Gödel's mathematical Platonism [1964] by much more than a change of emphasis. In my discussion, I shall take physical realism for granted; that is, I shall *call* real the things we handle, eat, perform physical experiments with, etc., and whatever can be said to interact with them, and I shall *profess* that mathematical physics has much to teach us concerning real things. In agreement with recent formulations of physical realism [cf. Armstrong 1978; also Bunge 1974–, III, 104ff], I shall assume that the properties and relations of physical things are no less real than the things themselves.

If someone says that we feign that there are constructs, to which our mathematical statements refer, he will most naturally be understood to mean that we produce objects in our fancy, which thereafter constitute definite, stable, albeit ghostly referents of our discourse. It is in this sense that some atheists maintain that there is a fantastic being, created by men in their own likeness, to whom the majority of mankind refer in their persistent talk of God. It is in this same sense that literary authors are usually said to create the characters about which they write in their books. However, the idea of creation, even if only of fictions, is wrought with such ontological difficulties that to many philosophers the ordinary under-

standing of fiction has seemed untenable. Thus, with regard to God, they would have us choose between Descartes' argument that God must exist, given that we can think of Him, and the positivist tenet that theology is nonsense. Likewise, they find it difficult to accept that such phrases as 'Pegasus, the flying horse' or 'the Mad Hatter's favourite tea blend' might refer to anything, fictitious or not. They prefer therefore to understand the feigning of referents for these and other kinds of discourse on the analogy of the feigning of feelings and actions. Just as one can feign sorrow while inwardly rejoicing, or feign selling a property one is merely placing under someone else's name, surely one can feign that he is referring to something when there is nothing in fact he could be speaking about. An understanding of mathematical discourse along some such lines has had many adherents in the twentieth century. Bunge, however, is not one of them. He dismisses them as "the literalists," who will not accept what to everybody else is clear, namely, "that mathematical symbols designate mathematical constructs" [1974–, II, 166]. There is still a third conceivable style of fictitious reference, illustrated by the familiar case where one actually refers to something real, but feigns it is other than it is. Thus, in fleet maneuvers the artillery officer commands his men to shoot at the enemy position, which is actually an empty beach. Under a dictatorship, a writer barely disguises his criticism of the government by telling a tale seemingly about a Babylonian king. When brought to bear on mathematical discourse these three senses of feigning yield three versions of fictionalism, which I shall call fictionalism$_1$, fictionalism$_2$, and fictionalism$_3$, respectively.

Let us first consider fictionalism$_3$. This appears to be a very suitable framework for describing the common procedures of applied mathematics. When asked to calculate the period of a given pendulum at a given place we feign that the pendulum hangs from a weightless, inextensible string, and that the sinus of the angle of displacement is equal to the angle itself, and rounding up the values of the pendulum's length and the local acceleration of gravity, we compute 2π times the square root of their quotient to an agreed decimal. From our fictitious assumptions we derive a result which is admittedly false, but which will differ from the measured values of the period by less than a prescribed number. At a less modest level, the cosmologist feigns that matter is homogeneously distributed in the universe and figures out the broad patterns of its evolution, although he knows all too well that matter is not equally distributed inside a pulsar and in intergalactic space. As the saying goes: one 'idealizes' reality in

order to obtain an approximate yet manageable picture of it. Turning to pure mathematics, I find that fictionalism$_3$ may have inspired some of the mathematicians who in the nineteenth century carried out the arithmetization of analysis. As a result of their labors anyone speaking, say, of a complex number could henceforth be regarded as actually referring not to some new-fangled esoteric individual entity, but 'merely' to an ordered pair of infinite classes of classes of classes of natural numbers. But then, of course, a truly hardnosed fictionalist would observe that even the natural numbers do not really exist, at any rate, not the larger ones, nor the so-called set of them. Fictionalism$_3$ was probably also at the back of Hilbert's mind when he proposed to show that the mathematical infinite was just a manner of speaking, something to which the mathematician merely seemed to refer, when he was really reasoning only about the finite. On the other hand, the actual execution of his program for proving that this is so suggests rather a fictionalist$_2$ approach to classical mathematics – which is viewed as a lawful system of strings of meaningless symbols – coupled with a downright realist view of the new discipline of metamathematics, the theory of such lawful systems. Anyway, the impossibility of Hilbert's program makes the exact classification of his position less important to us than it would otherwise be. I have mentioned him only because, as I hope to make clear in the sequel, that very impossibility implies that fictionalism$_3$ is not a viable philosophy of classical mathematics.

The only passage I have found where Bunge appears to countenance fictionalism$_3$ is not in the main text of his *Treatise* but in a table of the chief views on signs and constructs [1974–, I, 31]. In the line summarizing his own position he enters the following equation:

Construct = Equivalence class of brain processes.

I take it to mean that this is what constructs, and hence mathematical objects, *really are*, though we *feign* them to be all sorts of things, quaternions, ultrafilters, Lie groups, inaccessible cardinals, and what not. Now, in the light of what we know about the structure of matter at the level relevant to brain processes, it should be evident that the number of significantly distinct brain states, including *possible* brain states and even making allowance for genetic mutations, must be less than some very large but finite integer. Since each kind of brain process is a transition or a finite sequence of transitions between brain states, their number is finite too. This means that all the mathematics that can be built by taking equiva-

lence classes of brain processes – whether one feigns or not that they are something else – is comprised within combinatorics. Hence the undeniable philosophical importance of Hilbert's attempt to recreate Cantor's Paradise in a combinatorial setting, so to speak as a wall-paper garden or *coulisse*. Hence too the importance of Gödel's proof that not even elementary (i.e., non-analytic) number theory will fit into such a narrow setting. Generally speaking, it should be clear that not only the different kinds of human brain processes, but every kind of physical object available to man, will not suffice to provide all the objects that classical mathematics appears to speak about. Therefore fictionalism$_3$ is not a reasonable account of classical mathematical discourse. Consequently, if its objects do not exist of themselves as Platonic *noetà* – as Gödel himself believed – or in the mind of God, then *either* they are feigned through and through by the creative powers of the human intellect, which, to achieve this, must indeed be formidable, *or* mathematical discourse actually lacks referents and is a case of feigned referring. These are the theses I labelled fictionalism$_1$ and fictionalism$_2$, respectively. Let us take them up in this order.

Before going further into the matter I wish to remark that if mathematical objects are fictitious throughout, as fictionalism$_1$ maintains, there is no need to feign besides that they are brain processes, especially if one will end up by feigning that they are not. If we can feign them at all, then we can feign them to be what we want them to be, namely the elements, properties or relations of this or that mathematical structure, altogether clean of every psychic or encephalic connotation. This is not to deny that to think of them is indeed a mental process. But one must not confuse the act of feigning with what is feigned by it. The latter not only outlives the former, but may also be arrived at in many different ways. Thus, though in the present state of neurology we can say very little about such things, it does not seem at all likely that there was a great resemblance between, say, the brain processes of Ricci-Curbastro and those of Elie Cartan, when they worked on the development of modern differential geometry. Moreover, in stark contrast with the objects of ordinary imagination, which, according to some psychologists, can exhibit no more features than the act of imagining them bestows on them [cf. Sartre 1940], the fruits of mathematical fiction usually harbor surprises their creators never dreamt of. Suppose, for instance, that we feign – as indeed, according to fictionalism$_1$, someone must once have feigned – a standard model of first order arithmetic. For greater precision, let us assume that the model is ω, the set of finite ordinals, with addition and multiplication defined in the usual

way. (The successor Sn of a finite ordinal n is $n \cup \{n\}$. The empty set $\emptyset$ is the only finite ordinal which is not the successor of another one, and may be denoted by 0. If m and n are finite ordinals, $m + 0 = m$; $m + Sn = S(m + n)$; $m \cdot 0 = 0$; $m \cdot Sn = (m \cdot n) + m$.) Given ω, one can build out of it a first order language, some subset of whose sentences is the axiomatic theory of first order arithmetic. (Take any system of Gödelization of first order syntax and identify the Gödel numbers with the expressions they stand for.) It can then be shown that if this theory has any model at all – and ω itself was contrived just for this role, in our hypothesis – it possesses a denumerable non-standard model, that is to say, a denumerable model not isomorphic with ω. (Let $\langle M, 0, S, +, \cdot \rangle$ be such a model; there are then many bijective mappings of ω onto M, but none of them are structure-preserving.) Thus by the very act of feigning our intended standard model of arithmetic we conjure up an unintended model. There is no magic in this, for the unintended model is denumerable and hence can be drawn entirely from the domain of the intended one. But it does confirm what I remarked about the undreamt-of surprises concealed in mathematical fiction, for, as is well known, non-standard models of arithmetic exhibit some shocking features. Let us say that an element a stands in *the line of succession* of an element b if a can be reached from b by compounding the successor function a finite number of times. (I.e., if $a = SS \ldots Sb = S^n b$.) Two elements will be said to belong to the same *dynasty* if they are identical or if one stands in the line of succession of the other. It is clear that belonging to the same dynasty is an equivalence which partitions any model of arithmetic into equivalence classes or dynasties. But while a standard model consists of single dynasty, a denumerable non-standard model comprises infinitely many of them, each of which has infinitely many members. Only one of them, the dynasty of zero, is a submodel isomorphic with ω, and hence a standard model. Every model of arithmetic is totally ordered by the relation '$<$' ($a < b$ if and only if $(Ex)(a + Sx = b)$). This order induces a total order in the set of dynasties of a denumerable non-standard model: one dynasty is less than another if a representative of the former is less than a representative of the latter. (It can be shown that this relation does not depend on the choice of the representatives.) The set of dynasties is dense in this order: if F and G are dynasties such that $F < G$, there is a dynasty H such that $F < H < G$. If fictionalism$_1$ is right, whoever first devised a standard model of arithmetic unwittingly feigned this monster as well.[5]

A different example may give us a firmer grasp of what I dare to call the

substantive objectivity of mathematical structures. I propose to consider the structure underlying the game of chess. We shall view it as a finite directed graph,[6] the Graph of Chess, which can be constructed as follows. First take all injections of the 32 pieces of chess or of any subset thereof into the 64-square chessboard. Identify any two injections that can be obtained from one another by interchanging the values (i.e., the positions) of two equal pieces of the same color. The objects thus obtained we call *points*. In particular, the injection that yields the standard initial position of the game we call the starting-point. Take then every conceivable legal chess move. Each can be said to join a pair of points, namely, the point yielding the position at which the move begins and the point yielding the position at which the move ends. We call each such move an *edge*, which we regard as directed from the first point towards the second. We call an edge 'white' if it is a move of the whites; otherwise we call it 'black'. A path consisting of alternately black and white edges will be called a chequered path. The graph thus obtained is certainly finite, but it contains much more than we need. Thus, the injections of sets of pieces that do not contain both kings are isolated points of the graph which we may just as well ignore. But the Graph of Chess we are looking for is a subgraph of the graph we have built. It consists of (i) the starting-point and every point that can be reached from it by chequered paths whose first edge is white; and (ii) all the edges included in such paths. It is plain that this subgraph underlies the practice of the game and provides the necessary and sufficient basis for any theoretical study of it. The Graph of Chess is a definite, substantive entity. It can be embodied in an intricate fishing-net, with strings painted black and white and with the direction along each string given by the sense in which its threads are twisted. Indeed the graph is quite independent of the game itself. Except for the coloring of the edges and the privileged position of the starting-point, we may forget all about the game and consider the graph on its own. (In fact, it might even be unnecessary to distinguish the starting-point, if, as it is likely, it can be singled out by its very location in the graph – but I am not sure about this.) The different distributions of chess pieces on the chessboard can then be regarded as code-names for the points of this graph. As is often the case with code-names, they do not merely label the points but store information about them, specifically about their connections in the graph. Now, we will probably all agree that chess is a human invention. But was the Graph of Chess *created* by the inventor of the game? Should we not rather say that he *discovered* it? The Indian mathematician who, according to the

legend, charged his king $2^{64} - 1$ grains of wheat for the invention of chess, probably had no inkling of graph theory. Yet he lighted on the marvelous idea of turning a fixed mathematical structure, which we now know can be aptly conceived as a graph, but of which nobody has ever had a complete view, into the setting of a game, in which two players, exploring the structure's labyrinthine paths, choosing by turns at each crossing which way to follow, seek to drive one another into a blind alley. Since the Graph of Chess, like every finite graph, is isomorphic with a graph in space, it is the structure of a possible physical object – such as the fishing-net I mentioned earlier – and it is therefore a substantial universal, in Bunge's sense [1974-, III, 105]. But if, as fictionalism$_1$ maintains, classical mathematical discourse actually succeeds in referring to the structures it appears to speak about, then these structures are not a whit less objective and substantive than the Graph of Chess, and they are comprised, together with it, in the total furniture of the universe.[7] Whether they are Platonic *eidoi*, or divine thoughts, or figments of human fancy, is but of little consequence in this regard. For the fictionalist$_1$ who upholds the last alternative thereby simply ascribes to men the power to bring forth and sustain precisely the same kind of determinate, changeless, invisible and untouchable, yet thoroughly intelligible entities that Plato placed in the *hyperouranios topos* and his medieval disciples in God's understanding. The metaphysical problems generated by the admission of such entities are not lessened but rather heightened by the fictionalist's choice, for his philosophy must not only account for the existence of the Platonic realm but must also trace its origin right down to us humble creatures. It looks therefore as if the only way of avoiding these difficulties and yet retaining classical mathematics were that of fictionalism$_2$. No wonder then that so many have followed it in our un-Platonic century.

Some of the strongest pronouncements is favor of fictionalism$_2$ are to be found in Wittgenstein's posthumous *Bemerkungen über die Grundlagen der Mathematik* [1964]. Thus, one could hardly think of a more uncompromising statement of this position than the following 'explication' of the term *ideal object*:

> 'The sign *a* designates an ideal object' should obviously say something about the meaning and hence about the use of *a*. And it says of course that this use is in a certain respect similar to that of a sign that has an object, *and* that it does not designate an object. [Wittgenstein 1964, p. 136]

Coupled with Wittgenstein's radical conventionalism in logic, his con-

ception of mathematics as a "motley of techniques of proof" [1964, p. 84], to be learned by acquiring "conditioned calculating reflexes" [1964, p. 96] implies that mathematical discourse is a parrot's chatter, that not merely lacks reference, but is also empty of all thought. Indeed, in his opinion, the latter property must go with the former, for, as he puts it,

> What does it mean to obtain a new concept of the surface of a sphere? How is it then a concept of the surface of a *sphere*? Only in so far as it can be applied to real spheres. [1964, p. 134]

Wittgenstein's vision is, to my mind, so far removed from the reality of mathematics, except perhaps as one might encounter it in some dreary high-school or freshman course, that I do not think one ought to center upon this vision a discussion of fictionalism$_2$. I turn therefore to a more sensible version of the latter, which is sometimes disparagingly referred to as 'if-thenism' [see Musgrave 1977]. I take it that Rudolf Carnap adopted some such view after meeting Tarski in 1935. The if-thenist philosopher acknowledges the absolute value of logic and conceives of mathematical statements as logically true implication statements. E.g., 'If an object exemplifies such and such a structure, then it possesses this or that feature'. There need not be any such object for the statement to be true and interesting. Take a specific example. If in Hilbert's axioms of geometry we let X, Y and Z stand for the three arbitrary sets Hilbert calls 'points', 'lines' and 'planes', and we allow the letters R_1, R_2, R_3, R_4, R_5, to represent the five primitive relations of incidence (2), betweenness and congruence (2), we may define a Euclidean space as an octuple $\langle X, Y, Z, R_1, R_2, R_3, R_4, R_5\rangle$ that satisfies Hilbert's axioms. Then, the statement that, if S is a Euclidean space, the sum of the interior angles formed by three lines in S that mutually intersect by pairs is equal to two right angles, is true and significant, even if no Euclidean space exists at all and the terms 'S', 'line', 'angle', lack a referent. Indeed, the truth of the statement would be beyond all question if no Euclidean space could possibly exist due to the inconsistency of Hilbert's axioms, though in such a case we would not regard the statement itself as being particularly significant. This last remark suffices to explain what Wittgenstein somewhat irresponsibly called "the superstitious fear and awe of mathematicians in face of contradiction" [1964, p. 53]. They are not afraid of being wrong but they are afraid of being trifling. Unfortunately, if-thenism is not in a very good position to allay those fears. For, barring the simpler mathematical theories that can be translated into sound and complete

formal systems in which it is syntactically impossible to derive a contradiction, most mathematical conceptions can only be proved consistent by producing a model of them, which, of course, if it is not a physical object, must be either a Platonic entity or a substantive fiction, in the sense of fictionalism$_1$. (Alternatively, one may produce a model for another, more likely conception, with the aid of which then a formal system spelling out the conception being discussed can be shown to be syntactically consistent.) This circumstance, however, though it may make if-thenism unpopular with working mathematicians, does not constitute a refutation of it. Pure mathematics can be carried on while the researcher blindly trusts that the theory he is busy with is satisfiable, even if there are no real or ideal or fictitious entities presently available to satisfy it. If the mathematician's faith turns out to be wrong, that will not ruin the truth of his conclusions, but only their relevance. The following difficulty is harder to overcome from the if-thenist point of view. The property of logical truth and the relation of logical consequence, which, if if-thenism is right, are all that mathematics is really about, have so far been explained satisfactorily, with the breadth and scope required by classical mathematics, only within the framework of Tarskian semantics. Now Tarskian semantics resorts to a naive understanding of the set-theoretical predicates 'to be a set' and 'to be a member of' which can only be substantiated by providing a model of a sufficiently rich version of set theory. In the particular case of these basic concepts, on which the meaningfulness of all further mathematical concept-building is made to rest by if-thenism, one ought therefore to say after Wittgenstein: "How are they then concepts of a *set*, of *set-membership*? Only in so far as they can be applied to real sets." It could also be argued against the if-thenist version of fictionalism$_2$ that the methods of proving universal statements of implication in mathematics normally involve instantiation, whereby reference is actually made to some mathematical object, either real, ideal, or fictitious. But I shall not press these points, for I find that the insufficiency of if-thenist fictionalism$_2$ can be laid bare once and for all by reflecting on some well-known methods of mathematical physics.

For the if-thenist it is only in applied mathematics that the whole enterprise of mathematicians finally comes to fruition. The antecedents of some of the implications proved here at last acquire referents of which they, and hence also the respective consequents, are true. On our assumption of physical realism – which, by the way, Carnap did not share – the

statements of applied mathematics must be regarded as fully interpreted, at least in so far as they are concerned with reality. What matters to us now is that not all of them are exclusively concerned with it. In the cases I have in mind, which include the familiar applications of the calculus of variations in classical mechanics and in other branches of physics, a real physical object – a thing, state, process, etc. – is considered together with a multitude of unreal physical objects of the same kind and incorporated with the latter into a mathematical structure, the intrinsic properties of which are then used to single out the real object among the unreal ones (e.g., because it is the only path in the structure on which a certain integral takes an extremal value, etc.). I contend that if-thenist fictionalism$_2$ cannot account for cases like these. For suppose the if-thenist philosopher tried the usual gambit and claimed that the applied mathematician is not actually referring to the structure in question, consisting of a unique real and a multitude of unreal elements, but is merely stating the following logical true implication, which does not involve such a reference: '*If* such-and-such a structure existed, *then* it would contain those objects and the real one would hold such-and-such a peculiar position amidst the unreal ones'. In claiming this, the if-thenist would plainly misconstrue the applied mathematician's argument, the very upshot of which is that, on his assumptions, only the real object can actually exist, while all its near relatives within the structure are in effect impossible. The applied mathematician cannot therefore be understood to mean that if the structure considered by him did exist such and such consequences would ensue, for he is saying at the same time that the said structure cannot really exist at all. But in order to say so much he must succeed in referring to that structure and to the whole bunch of its unreal elements. He cannot just feign that he is referring to them. We may readily grant, indeed, that he does not actually refer to the real object, or, at any rate, that he does not refer to it directly, but only through an ideal or fictitious representative of it, which, as such, finds its natural place in the neighborhood of the unreal elements of the structure. (Thus the actual motions of a complex mechanical system can be represented by a simple path in configuration space.) But to that representative he must perforce refer, for it is only by being referred to that it can come to stand for anything. Here we face, therefore, a family of cases in which either there is no sense at all or actual reference is being made knowingly to unreal objects. It is clear that such cases can only be accounted for by Platonism – which

bestows a meta-physical reality on the physically unreal objects referred to – or by fictionalism$_1$, and that they lie altogether beyond the reach of fictionalism$_2$.

The following example, drawn from S.W. Hawking and G.F.R. Ellis' book, *The Large-Scale Structure of Space-Time* [1973], will illustrate the point I want to make. It will also throw light on the relationship between this procedure of referring, in mathematical physics, to acknowledgedly fictional or ideal objects, and the procedure we described earlier, when talking about fictionalism$_3$, by which a real physical object is feigned to be other than it is and is pictured in an 'idealized' form. Hawking and Ellis assume that the physical world is or can to a good approximation be represented by a real four-dimensional non-compact Hausdorff C^∞-differentiable manifold, endowed with a Lorentz metric and with further geometrical objects corresponding to its diverse physical properties. Such a manifold is called a *spacetime*. A Lorentz metric on a spacetime M partitions all the tangent vectors at each point of M into the three mutually exclusive classes of spacelike, null and timelike vectors. A curve in M is said to be timelike if all vectors tangent to it are timelike; null, if all vectors tangent to it are non-zero and null; and non-spacelike if all its tangent vectors are either timelike or null and non-zero. Any spacetime is supposed to admit an everywhere timelike vector field, that makes it possible to classify all non-spacelike vectors at each point into future-directed and past-directed vectors. This classification applies also to the curves to which such vectors are tangent. It is also assumed that all transfers of energy and momentum take place along non-spacelike curves. In other words, all signals, all causal actions propagate along such curves. The authors assume moreover that the Lorentz world metric is bound to the distribution of matter by the Einstein field equations. Now, Gödel has shown that a spacetime whose metric is governed by the Einstein field equations can under certain conditions contain closed timelike curves [1949]. Hawking and Ellis regard the existence of such curves in the spacetime representing the physical world as paradoxical,

> for one could imagine that with a suitable rocketship one could travel round such a curve and, arriving back before one's departure, one could prevent oneself from setting out in the first place. Of course there is a contradiction only if one assumes a simple notion of free will; but this is not something which can be dropped lightly since the whole of our philosophy of science is based on the assumption that one is free to perform any experiment. [1973, p.189].

To avoid such paradoxes the authors assume that the world fulfils three

further conditions, namely, the *chronology condition*, that there are no closed timelike curves; the *causality condition*, that there are no closed non-spacelike curves; and the *strong causality condition*, which holds in a spacetime M if at every point P in M, every neighborhood of P contains a neighborhood of P which is not traversed by any given non-spacelike curve more than once. However, even the strong causality condition does not completely rule out all causal paradoxes, for a spacetime can be, so to speak, on the verge of violating the chronology condition in that the slightest variation of the metric would generate closed timelike curves. Hawking and Ellis do not countenance such a situation as physically viable because Einstein's Theory of Gravitation is "presumably the classical limit of some, as yet unknown, quantum theory of space-time and in such a theory the Uncertainty Principle would prevent the metric from having an exact value at every point" [1973, p. 197]. This difficulty can be overcome by assuming that the chronology condition is *stable*, in the sense that it is shared by all spacetimes that differ but slightly from the real one, and are, so to speak, its neighbors. In order to give an exact meaning to this notion of neighboring spacetimes one must define a topology on the set of all spacetimes. The authors do not tackle the problem of uniting in one connected topological space spacetimes with incompatible topologies – which, according to them, can be done – but they only consider the definition of a specific topology on any set of diffeomorphic spacetimes that differ only in their Lorentz metrics. We need not worry about the details of their definition [1973, p. 198]. It is enough to mention that a given spacetime is said to meet the *stable causality condition* if it has an open neighborhood in this topology such that every spacetime belonging to that neighborhood fulfils the chronology condition. The authors prove the following, cosmologically significant, theorem: A spacetime M meets the stable causality condition if and only if there is a real-valued function on M whose gradient is everywhere timelike. Any such function can be viewed as a cosmic time function, for either it or its product by minus one increases monotonically along every future-directed non-spacelike curve. On the other hand, any reasonable cosmic time function must have an everywhere timelike gradient. The authors have therefore succeeded in determining the necessary and sufficient condition for the existence of a cosmic time by means of a notion, the condition of stable causality, which can only be conceived of by inserting the real world, or if you please, its ideal or fictional mathematical representative, into a non-denumerable set of unreal worlds that resemble it, and by regarding them all as the points

of a connected topological space. It is plain that this construction can only make sense if one actually succeeds in referring to the space of spacetimes and to its several points. It will be noticed, moreover, that Hawking and Ellis' move looks a good deal bolder that the familiar talk of possible worlds, say, in formal semantics, since the several spacetimes are not just collected in a set but are tightly knit together in a topological structure. And yet a short reflection should persuade us that some such move is implicit in the trite method of idealization we discussed at the outset, in connection with fictionalism$_3$. In that method one 'approximates' a real situation by an unreal one that is mathematically more manageable. Now the concept of approximation can only be given a clear meaning in a topological context. To view a given ideal physical system A as an approximate representation of the real system B is tantamount to conceiving of A and B as neighboring points in a suitable topological space. Such a conception is always tacitly but quite definitely implied in the mathematical handling of even the simplest physical situations. In the more familiar cases, one can approximate the real B by the ideal A only in as much as both systems are viewed as determined by the same parameters $x_1, \ldots, x_n$, which take the definite real values $a_1, \ldots, a_n$ at A, while taking at B other, indefinite, but neighboring values $b_1, \ldots, b_n$ (with $a_i - \sigma_i < b_i < a_i + \sigma_i$ for some small positive real number σ_i; $1 \leq i \leq n$). A and B are thereby treated as neighboring points in $\mathbf{R}^n$. In other, less simple cases, the topological space involved is more unusual—as in the above example from relativistic cosmology—but no less essential. Since approximation by idealization has been up to now the surest, indeed the only unquestionably successful way of increasing our understanding of the world, one might venture to conclude that the intellection of real things can only be achieved by finding the right place for them in a setting of *intelligibilia*. Whether the latter subsist eternally, as Platonists teach, or are man-made, as fictionalism$_1$ would have it, is not, I dare say, a question of much genuine philosophical moment. Indeed, shifting from one of these two views to the other does not imply or demand a greater change in the experienced content of the matter than does, say, a *Gestalt* switch. The foregoing discussion suggests a final remark regarding the claim, apparently made by Bunge, that real things do not actually possess 'mathematical features'. (See the quotation from [1974–, III, 118] on p. 400 above.) For aught I know, this claim may be correct, but unless it is wrong one cannot meaningfully contend that mathematical physics provides an approximate representation of physical truth, let alone that it converges to it; for, as we have seen,

a mathematical object A can be said to approximate an object B only if the latter belongs together with the former in the same mathematical structure.

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NOTES

* A slightly different version of this paper was read, under the title 'Mathematics, Fictionalism and Ontology', at the meeting of the Society for Exact Philosophy held in Montreal on June 4–5, 1979.

** "God knows whether or not there is a Dulcinea in this world or if she is a fanciful creation. This is not one of those cases where you can prove a thing conclusively. I have not begotten or given birth to my lady, although I contemplate her as she needs must be, seeing that she is a damsel who possesses all those qualities that may render her famous in all parts of the world ..." (Cervantes 1949, p. 723).

[1] I refer specifically to volume 3 of his [1974–], 7 vols., of which the first three had appeared when this was written.

[2] Kant, *Untersuchungen über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral* (1764); *Kritik der reinen Vernunft*, Transzendentale Methodenlehre, I. Die Disziplin der reinen Vernunft (i) im dogmatischen Gebrauche, (iv) in Ansehung ihrer Beweise.

[3] See, in particular, the logico-ontological calculus developed in Leibniz [1890], pp. 236–247.

[4] On the meaning and purpose of Poincaré's conventionalism, see my [1978], pp. 320ff.

[5] For a proof of the above statements on non-standard models of arithmetic, see Boolos and Jeffrey [1974], pp. 192ff.

[6] We define a *directed graph* to be a pair $\langle V, E\rangle$, where V is any set and E is a subset of $V \times V$. The elements of V are the *points* or *vertices* of the graph; the elements of E are the *edges* of the graph. If $k = \langle x, y\rangle$ is an edge, k is said to *join* the points x and y, which are called the *endpoints* of k. A graph $G' = \langle V', E'\rangle$ is a subgraph of the graph $G = \langle V, E\rangle$ if $V' \subset V$ and $E' \subset E$. A *path* in a graph G is a sequence $k_1, k_2, \ldots, k_n$ of edges of G, such that, for every integer i ($1 \leq i < n$), the second endpoint of k_i equals the first endpoint of k_{i+1}.

[7] For a forceful defense of fictionalism$_1$ from a working mathematician's standpoint, see R. Hersh [1979].

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